Int. J. Heat Mass Transfer. Vol. 26, No. 6, pp. 861-869, 1983
Printed in Great Britain

0017-9310/83/060861-09 $03.00/0
Pergamon Press Ltd.

EFFECTIVE MACROSCOPIC DESCRIPTION FOR HEAT CONDUCTION IN PERIODIC COMPOSITES

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(Received 22 May 1982)

Abstract—Effective parameters for the macroscopic behaviour of periodic composites are determined, concerning static or transient heat conduction, when the wavelength is large compared to the length of the period. Different situations are analysed using the homogenization method, which lead to different macroscopic descriptions: single classical partial differential equations, single integral partial differential equations with memory effect, systems of partial differential equations, etc. Some simple examples are given where analytical results are possible.

NOMENCLATURE

- h, length of the period for laminated composites;
- k, particular solution for T;
- l, characteristic microscopic length;
- $\mathbf{n}$, unit normal to Γ;
- n, porosity;
- t_i, particular solution for T;
- $\mathbf{x}$, low space variable;
- $\mathbf{y}$, fast space variable;
- C, thermal capacity;
- $\hat{K}$, memory function;
- L, characteristic macroscopic length;
- T, temperature;
- $\mathscr{V}, \mathscr{W}$, functional spaces with functions θ.

Greek symbols

- α, diffusivity;
- β, dimensionless pulsation;
- δ, Kronecker symbol;
- ε, small parameter;
- λ, conductivity;
- ω, pulsation;
- ρ, volumic mass;
- τ, dimensionless time;
- Γ, boundary between two media;
- Ω, spatial period.

1. INTRODUCTION

INTEREST in the development of continuum macroscopic models for composite materials is not new and many papers have been devoted to this subject. Most of them are dedicated to laminated composites since in this case the geometry allows analytical results [1, 6]. More sophisticated situations [7, 9] are less usual and experimental data quite rare [10, 11].

The macroscopic heat transfer in periodic composites is studied here using the homogenization method [12, 14]. The aim of the paper is to determine parameters equivalent to thermal conductivity and capacity, and to verify the macroscopic differential system satisfied by temperature for periodic composites.

At first, in Section 2, the asymptotic method of homogenization is presented. This method uses a small parameter which measures the characteristic length of the period (i.e. of the heterogeneities) compared to a macroscopic length.

In Section 3, the classical case for static and transient heat transfer is considered. The procedure is quite similar to that described in refs. [12, 15]. The macroscopic behaviour is generated by a single partial differential equation of classical structure. The method gives the effective parameters and their properties are studied. As an example, the case for a bilaminated composite is investigated and the well-known results for this particular case are obtained.

Section 4 is devoted to composites with strong discontinuities in conductivity. Pulsation dependent capacity is demonstrated and the macroscopic behaviour is shown when transient heat transfer occurs. Some simple examples are given where analytical results are possible. The memory effects displayed here are of similar character to those for saturated porous media [16, 17, 19].

In Section 5, the special case of a trilaminated composite is presented which exhibits a system of two coupled integro-differential equations with two fields of temperature.

2. FORMULATION OF THE PROBLEM: HOMOGENIZATION

We consider a composite with a fine periodic structure. The period Ω, of dimension $O(l)$, is small compared with the characteristic length L of the medium at the macroscopic scale,

$$\varepsilon = \frac{l}{L} \ll 1.$$

For simplicity we assume the medium to be composed of two or three solids distinguished by the subscripts 1, 2 and 3, occupying the domains Ω_1, Ω_2 and

Ω_3, respectively. The results can be easily extended to p arbitrary solids. The boundary between two media will be denoted by Γ. (See Fig. 1 for the case of a binary system.)

At the initial time, the medium is in thermal equilibrium and the temperature has a constant value throughout the period. We consider a perturbation from this equilibrium, with pulsation ω, in such a way that the wavelength is large compared to the characteristic length l of the period. In the following, the temperature increment is given by

$$T(\mathrm{x}) \exp(i\omega t)$$

where T is a function of the space coordinates $\mathrm{x} = (x_1, x_2, x_3)$. The following study includes as a particular case the static situation where the pulsation is zero.

The determination of the macroscopic laws for heat transfer, that is the behaviour of an equivalent continuous medium, is performed using the homogenization method [12, 13] based on an asymptotic expansion in powers of the small parameter ε and including a double scale with characteristic lengths l and L. It is assumed that the temperature T can be written as a function of two variables, $T(\mathrm{x}, \mathrm{y})$. The variable x is the macroscopic space variable and $\mathrm{y} = \mathrm{x}/\varepsilon$ the microscopic one, describing the small heterogeneities. The temperature T has the form

$$T(\mathrm{x},\mathrm{y}) = T^0(\mathrm{x},\mathrm{y}) + \varepsilon T^1(\mathrm{x},\mathrm{y}) + \varepsilon^2 T^2(\mathrm{x},\mathrm{y}) + \ldots \quad (1)$$

where the T^i are periodic in y with period Ω/ε.

In the following, Ω/ε is replaced by Ω since confusion is not possible. The method consists of incorporating an expansion of this type into the set of equations which describes the phenomenon and identifying the powers in ε, while keeping in mind the fact that x and y should be considered as independent variables and that the operator $\partial/\partial x_i$ is now expressed by

$$\frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i} = \nabla x_i + \frac{1}{\varepsilon}\nabla y_i, \quad i = 1,2,3.$$

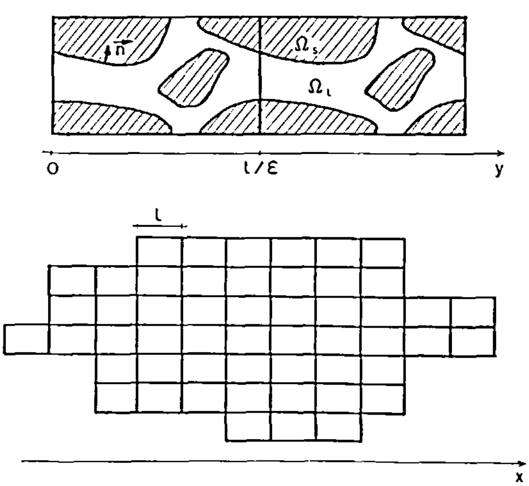

FIG. 1. Binary periodic composite.

The homogenization process produces a set of equations satisfied by T^0, which in fact represents the macroscopic behaviour $\varepsilon \mapsto 0$.

3. HEAT TRANSFER BY CONDUCTION IN A BINARY COMPOSITE: CLASSICAL CASE

The set of equations giving the temperature T is

$$\nabla(\lambda_i \nabla T_i) = \rho_i C_i \, i\omega T_i, \quad i = 1,2, \qquad (2)$$

with, on the boundary Γ between the two components,

$$[T]_\Gamma = 0, \qquad (3)$$

$$[\lambda \nabla T]_\Gamma \cdot \mathrm{n} = 0 \qquad (4)$$

where λ denotes the conductivity tensor. The tensor λ is definite positive and symmetrical, ρ is the volumic mass, C the thermal capacity at constant volume, $[\phi]_\Gamma$ the discontinuity of ϕ on Γ, and n a unit normal to Γ.

The quantities λ, T, ρ and C are *a priori* functions of x and y and take the values λ_i, T_i, ρ_i, C_i, with $i = 1, 2$ in media 1 and 2. In the classical case, all the different terms in equation (2) are assumed to be of the same order of magnitude for the macroscopic level

$$\rho C \omega L^2/|\lambda| = O(1).$$

All coefficients in equations (2)–(4) are of order 1. Expansion (1) will now be introduced into equations (2)–(4) as follows:

3.1. *Equation* (2) *for the order* ε^{-2}, *equation* (3) *for* ε^0 *and equation* (4) *for* ε^{-1}

$$\nabla_y(\lambda \nabla_y T^0) = 0,$$

$$[T^0]_\Gamma = 0$$

and

$$[\lambda \nabla_y T^0]_\Gamma \cdot \mathrm{n} = 0$$

where ∇_y stands for differentiation in the y space variable. It is straightforward to obtain the solution. Let $\mathscr{V}$ be the Hilbert space consisting of regular functions θ defined on Ω, which are Ω-periodic and satisfy the condition

$$\int_\Omega \theta \, d\Omega = 0,$$

with the scalar product

$$(\theta_1, \theta_2)_v = \int_\Omega \lambda \nabla_y \theta_1 \nabla_y \theta_2 \, d\Omega.$$

Multiplying equation (2) by $\theta \in \mathscr{V}$, integrating by parts and considering equation (4), gives an equivalent variational formulation determining T^0, to an arbitrary additive function of x introduced by the external condition

$$\int_\Omega \theta \, d\Omega = 0,$$

$$\int_\Omega \lambda \nabla_y T^0 \nabla_y \theta \, d\Omega = 0, \quad \forall \theta \in \mathscr{V}.$$

It follows, since λ is positive definite, that

$$T^0(\mathrm{x},\mathrm{y}) = T^0(\mathrm{x}), \quad \text{for arbitrary } T^0(\mathrm{x}).$$

Equation (3) implies that

$$T_1^0(\mathrm{x}) = T_2^0(\mathrm{x}) = T^0(\mathrm{x}).$$

3.2. *Equation (2) for the order ε^{-1}, equation (3) for ε^1 and equation (4) for ε^0*

$$\nabla_y(\lambda(\nabla_y T^1 + \nabla_x T^0)) = 0,$$
$$[T^1]_\Gamma = 0, \tag{5}$$
$$[\lambda(\nabla_y T^1 + \nabla_x T^0)]_\Gamma \cdot \mathrm{n} = 0.$$

Following the above procedure, the problem is equivalent to that given by equation (6), to an arbitrary additive function of x for T^1,

$$\int_\Omega \lambda \nabla_y T^1 \nabla_y \theta \, \mathrm{d}\Omega = -\int_\Omega \lambda \nabla_x T^0 \nabla_y \theta \, \mathrm{d}\Omega, \quad \forall \theta \in \mathcal{V}. \tag{6}$$

The existence and uniqueness of the solution result from the Lax–Milgram lemma. If $t_i(y)$ is the particular solution corresponding to

$$\nabla_{x_j} T^0 = \delta_{ij},$$

T^1 will be written as

$$T^1(\mathrm{x},\mathrm{y}) = t_i \nabla_{x_i} T^0 + \tilde{T}^1(\mathrm{x}), \quad \text{for arbitrary } \tilde{T}^1(\mathrm{x}).$$

The symmetry of LHS of equation (6), due to the symmetry of $\lambda (\lambda_{ij} = \lambda_{ji})$, enables us to write equation (7) after introducing in equation (6) $T^1 = t_i$, $\theta = t_j$ and then $T^1 = t_j$, $\theta = t_i$

$$\int_\Omega \lambda_{ih} \nabla_{y_h} t_j \, \mathrm{d}\Omega = \int_\Omega \lambda_{jh} \nabla_{y_h} t_i \, \mathrm{d}\Omega. \tag{7}$$

3.3. *Equation (2) for the order ε^0, and equation (4) for the order ε^1*

$$\nabla_y(\lambda(\nabla_y T^2 + \nabla_x T^1)) + \nabla_x(\lambda(\nabla_y T^1 + \nabla_x T^0)) = \rho C \, \mathrm{i}\omega T^0, \tag{8}$$

$$[\lambda(\nabla_y T^2 + \nabla_x T^1)]_\Gamma \cdot \mathrm{n} = 0. \tag{9}$$

Generally speaking, equations (8) and (9) do not admit any solution. The necessary and sufficient condition for the existence of a solution is a compatibility condition obtained by integrating equation (8) over Ω, using equation (9). The periodic property then leads to

$$\nabla_x(\langle\lambda\rangle \nabla_x T^0) = \langle \rho C \rangle \mathrm{i}\omega T^0$$

or for transient heat transfer

$$\nabla_x(\langle\lambda\rangle \nabla_x T^0) = \langle \rho C \rangle \frac{\partial T^0}{\partial t} \tag{10}$$

where

$$\langle\lambda\rangle = \frac{1}{|\Omega|} \int_\Omega \lambda(\delta + \nabla_y t) \, \mathrm{d}\Omega,$$

$$\langle\rho C\rangle = \frac{1}{|\Omega|} \int_\Omega \rho C \, \mathrm{d}\Omega,$$

are respectively the effective conductivity and the effective capacity of a macroscopic equivalent medium where heat transfer is described by equation (10). This formulation is valid from $\omega = 0$ (static case) until ω is such that $\rho C \omega L^2 = O(1)$. For greater values [$\omega = O(\varepsilon^{-1})$, for example] the process is inappropriate: the macroscopic length L, which is of the order of magnitude of the wavelength, becomes of the same order of magnitude as l: ε is no longer a small parameter and the periodic property of the solution falls down. The tensor $\langle\lambda\rangle$, ω-independent and symmetrical by virtue of equation (7), is positive definite and can be determined from equation (6). The effective capacity appears as the arithmetical mean-value of the capacity.

3.4. *Bilaminated composite*

Let us consider the particular geometry of a periodic layered medium (Fig. 2) consisting of two homogeneous media 1 and 2 occupying layers of respective thickness $(1-n)h/\varepsilon$ and nh/ε, measured with the space variable y; n denotes the partial volume of medium 2.

In this situation, periodicity is arbitrary along the y_2 direction (Fig. 2) and T must be independent of y_2. On the other hand, the medium exhibits a symmetry of revolution around the y_1 axis: directions 2 and 3 are equivalent. Let us solve the problem on the period

$$-(1-n)h/\varepsilon \leqslant y_1 \leqslant nh/\varepsilon.$$

The particular solution $t_1(y_1)$ of equations (5) with $\nabla_x T^0 = (1, 0, 0)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}y_1}\left(\lambda \frac{\mathrm{d}t_1}{\mathrm{d}y_1}\right) = 0,$$
$$[t_1]_\Gamma = 0,$$
$$\left[\lambda \frac{\mathrm{d}t_1}{\mathrm{d}y_1}\right]_\Gamma = 0,$$

t_1 h/ε periodic. Taking into account the condition

$$\int_\Omega t_1 \, \mathrm{d}\Omega = 0,$$

gives

$$t_1(y_1) = 0.$$

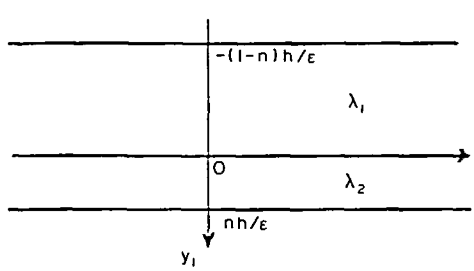

FIG. 2. Bilaminated composite.

For $t_2(y_1)$, the equations to be solved are

$$\frac{d}{dy_1}\left(\lambda \frac{dt_2}{dy_1}\right) = 0,$$

$$[t_2]_\Gamma = 0,$$

$$\left[\lambda\left(\frac{dt_2}{dy_1}+1\right)\right]_\Gamma = 0,$$

t_2 h/ε periodic, with zero mean-value. On the period, the solution is

$$t_2 = \frac{n(\lambda_2-\lambda_1)}{n\lambda_1+\lambda_2(1-n)} y_1 + D \quad \text{in } \Omega_1,$$

$$t_2 = \frac{-(1-n)(\lambda_2-\lambda_1)}{n\lambda_1+\lambda_2(1-n)} y_1 + D \quad \text{in } \Omega_2,$$

where D is a constant, the value of which can be disregarded in the following. The macroscopic heat conductivity tensor is expressed by

$$\langle\lambda\rangle_{ij} = \int_\Omega \lambda_{il}(\delta_{lj}+\nabla_{y_l} t_j)\, d\Omega,$$

which leads to

$$\langle\lambda\rangle_{12} = \langle\lambda\rangle_{13} = \langle\lambda\rangle_{23} = 0$$

and

$$\langle\lambda\rangle_{11} = \frac{\lambda_2\lambda_1}{n\lambda_1+(1-n)\lambda_2},$$

$$\langle\lambda\rangle_{22} = \langle\lambda\rangle_{33} = (1-n)\lambda_1 + n\lambda_2.$$

So we obtain the well-known classical result.

4. STRONG DISCONTINUITIES FOR CONDUCTIVITY IN A BINARY SYSTEM

We consider in this section the case where the conductivity in medium 2 is very small compared with that of medium 1, in such a way that both sides of equation (2) are of the same order of magnitude. Let

$$\lambda_2 = \lambda_2'\varepsilon^2, \quad |\lambda_2'| = O(1),$$

the other coefficients in equations (2)–(4) remaining $O(1)$. The case $|\lambda_2| = O(\varepsilon^p)$, $0 \leqslant p < 2$ is of no interest since for that situation results of Section 3 are valid. Equations (2)–(4) then become

$$\nabla(\lambda_1\nabla T_1) = \rho_1 C_1\, i\omega T_1 \quad \text{in } \Omega_1, \tag{11}$$

$$\nabla(\varepsilon^2\lambda_2'\nabla T_2) = \rho_2 C_2\, i\omega T_2 \quad \text{in } \Omega_2, \tag{12}$$

$$T_1 = T_2 \quad \text{on } \Gamma, \tag{13}$$

$$\lambda_1\nabla T_1 \mathbf{n}_1 = \varepsilon^2\lambda_2'\nabla T_2\mathbf{n}_2 \quad \text{on } \Gamma. \tag{14}$$

The homogenization process will now be applied.

4.1. *Equation* (11) *at* ε^{-2}, *and equation* (14) *at* ε^{-1}

$$\nabla_y(\lambda_1\nabla_y T_1^0) = 0,$$

$$\lambda_1\nabla_y T_1^0 \mathbf{n}_1 = 0 \quad \text{on } \Gamma.$$

The solution is

$$T_1^0 = T^0(\mathbf{x}).$$

The same equations for the orders ε^{-1} and ε^0, respectively, are expressed by

$$\nabla_y(\lambda_1(\nabla_y T_1^1 + \nabla_x T^0)) = 0,$$

$$\lambda_1(\nabla_y T_1^1 + \nabla_x T^0)\mathbf{n}_1 = 0 \quad \text{on } \Gamma.$$

The solution is obtained in the same way as in Section 3. If t_i is the solution with $\nabla_{x_j} T^0 = \delta_{ij}$,

$$T_1^1(\mathbf{x},\mathbf{y}) = t_i\nabla_{x_i} T^0 + \bar{T}^1(\mathbf{x}).$$

The solutions t_i satisfy a relation of the same type as equation (6) where integration is now restricted to Ω_1. From equations (12) and (13) at ε^0

$$\nabla_y(\lambda_2'\nabla_y T_2^0) = \rho_2 C_2\, i\omega T_2^0, \quad T_0^2 = T^0(\mathbf{x}) \text{ on } \Gamma, \tag{15}$$

where T_2^0 is Ω-periodic. With

$$T_2^0 = T^0(\mathbf{x}) + W$$

the boundary problem (15) becomes

$$\nabla_y(\lambda_2'\nabla_y W) = \rho_2 C_2\, i\omega(T^0 + W), \quad W = 0 \text{ on } \Gamma, \tag{16}$$

where W is Ω-periodic.

Let $\mathscr{W}$ be the space of regular functions θ with complex values, defined on Ω_2, Ω-periodic, of zero values on Γ, with the scalar product

$$(\theta_1, \theta_2)_{\mathscr{W}} = \int_{\Omega_2} (\nabla_y\theta_1\nabla_y\bar{\theta}_2 + \theta_1\bar{\theta}_2)\, d\Omega$$

where $\bar{\theta}_2$ denotes the complex conjugate. Multiplying by $\bar{\theta} \in \mathscr{W}$ on both sides of equation (16), integrating by parts and taking into account the periodicity, we obtain the equivalent variational formulation (17)

$$\int_{\Omega_2} (\lambda_2'\nabla_y W\nabla_y\theta^2 + i\omega\rho_2 C_2 W\bar{\theta})\, d\Omega = -\int_\Omega \rho_2 C_2\, i\omega T^0\bar{\theta}\, d\Omega. \tag{17}$$

The existence and uniqueness follow again from the Lax–Milgram lemma. The demonstration is very close to that given in ref. [20] for a vectorial unknown.

The solution can be written in the form

$$W(\mathbf{x},\mathbf{y}) = T_2^0(\mathbf{x},\mathbf{y}) - T^0(\mathbf{x}) = -k(\mathbf{x},\mathbf{y},\omega)T^0(\mathbf{x}) \tag{18}$$

where k is the complex valued and ω depending scalar solution of equation (17) with $T^0(\mathbf{x}) = -1$.

Equations (10) for the order ε^0 and (13) for the order ε^1 give

$$\nabla_y(\lambda_1(\nabla_y T_1^2 + \nabla_x T_1^1)) + \nabla_x(\lambda_1(\nabla_y T_1^1 + \nabla_x T^0)) = \rho_1 C_1\, i\omega T^0 \tag{19}$$

$$\lambda_1(\nabla_y T_1^2 + \nabla_x T_1^1)\mathbf{n}_1 = \lambda_2'\nabla_y T_2^0\mathbf{n}_1 \quad \text{on } \Gamma.$$

The compatibility condition is obtained nearly as in the classical case, by averaging equation (19). Using the

Ω-periodicity we have

$$\frac{1}{|\Omega|}\int_{\Omega_1}(\nabla_x\lambda_1(\nabla_y T_1^1+\nabla_x T^0)-\rho_1 C_1\,\mathrm{i}\omega T^0)\,\mathrm{d}\Omega$$
$$=\nabla_x\langle\lambda_1\rangle\nabla_x T^0-\langle\rho_1 C_1\rangle\,\mathrm{i}\omega T^0$$
$$=\frac{1}{|\Omega|}\int_\Gamma \lambda_1(\nabla_y T_1^2+\nabla_x T_1^1)\mathrm{n}_1\,\mathrm{d}S$$
$$=-\frac{1}{|\Omega|}\int_\Gamma \lambda_2'\nabla_y T_2^0\mathrm{n}_1\,\mathrm{d}S \tag{20}$$

where $\langle\lambda_1\rangle$ and $\langle\rho_1 C_1\rangle$ are defined by

$$\langle\lambda_1\rangle=\frac{1}{|\Omega|}\int_{\Omega_1}\lambda_1(\delta+\nabla_y\mathrm{t})\,\mathrm{d}\Omega,$$
$$\langle\rho_1 C_1\rangle=\frac{1}{|\Omega|}\int_{\Omega_1}\rho_1 C_1\,\mathrm{d}\Omega.$$

Considering equation (15) averaged on Ω_2 with again the Ω-periodicity, the last member of equation (20) becomes

$$-\frac{1}{|\Omega|}\int_\Gamma \lambda_2'\nabla_y T_2^0\mathrm{n}_1\,\mathrm{d}S=\frac{1}{|\Omega|}\int_{\Omega_2}\nabla_y(\lambda_2'\nabla_y T_2^0)\,\mathrm{d}\Omega$$
$$=\langle\rho_2 C_2\,\mathrm{i}\omega T_2^0\rangle.$$

Hence, using equation (18), we finally obtain the macroscopic behaviour as

$$\nabla_x(\langle\lambda_1\rangle\nabla_x T^0)=(\langle\rho C\rangle-\langle\rho_2 C_2 k\rangle)\,\mathrm{i}\omega T^0.$$

And since we have to the first order $\langle\lambda_1\rangle=\langle\lambda\rangle$ and the mean value of temperature given by

$$\langle T\rangle=T^0(1-\langle k\rangle),\quad \langle k\rangle=\frac{1}{|\Omega|}\int_{\Omega_2}k\,\mathrm{d}\Omega,$$

we can write, when $\varepsilon\mapsto 0$,

$$\nabla_x\left(\langle\lambda\rangle\nabla_x\frac{\langle T\rangle}{1-\langle k\rangle}\right)=(\langle\rho C\rangle-\langle\rho_2 C_2 k\rangle)\,\mathrm{i}\omega\frac{\langle T\rangle}{1-\langle k\rangle}. \tag{21}$$

It is clear that, just as in Section 3, $\langle\lambda\rangle$ is a symmetrical tensor. But now we lose the definite positive property: $\langle\lambda\rangle\nabla_x T\nabla_x T\geqslant 0$. Therefore equation (21) is only valid for those directions where $\langle\lambda\rangle\nabla_x T\neq 0$. If not, a periodic solution is impossible and scattering occurs.

Let us consider for simplicity the case where the λ_i, ρ_i, C_i are constants on Ω_i and do not depend on the macroscopic space variable x. Then equation (21) can be replaced by equation (22), which differs from the classical equation (10) in the complex and ω depending term $\rho_2 C_2\langle k\rangle$

$$\nabla_x(\langle\lambda\rangle\nabla_x\langle T\rangle)$$
$$=(\rho_1 C_1(1-n)+\rho_2 C_2 n-\rho_2 C_2\langle k\rangle)\,\mathrm{i}\omega\langle T\rangle. \tag{22}$$

It is clear from equation (15) that $k\mapsto 0$ when $\omega\mapsto 0$ and equation (22) [as equation (21)] is in this case similar to equation (10).

Introducing real and imaginary parts for $\langle k\rangle$,

$$\langle k\rangle=\langle k\rangle_1+\mathrm{i}\langle k\rangle_2,$$

and coming back to the time derivatives, the monochromatic macroscopic behaviour is described by

$$\nabla_x(\langle\lambda\rangle\nabla_x\langle T\rangle)$$
$$=(\rho_1 C_1(1-n)+\rho_2 C_2 n-\rho_2 C_2\langle k\rangle_1)\frac{\partial\langle T\rangle}{\partial t}$$
$$-\rho_2 C_2\frac{\langle k\rangle_2}{\omega}\frac{\partial^2\langle T\rangle}{\partial t^2}. \tag{23}$$

When a transient heating is applied to the composite, $\langle k\rangle$ must be replaced in equation (22) by a convolution operator with kernel $\hat{K}$ defined as the inverse Fourier transform of $\langle k\rangle/\mathrm{i}\omega$. We have

$$\nabla_x(\langle\lambda\rangle\nabla_x\langle T\rangle)=(\rho_1 C_1(1-n)+\rho_2 C_2 n)\frac{\partial\langle T\rangle}{\partial t}$$
$$-\rho_2 C_2\int_{-\infty}^{t}\hat{K}(t-\tau)\frac{\partial^2\langle T\rangle}{\partial\tau^2}(\tau)\,\mathrm{d}\tau.$$

4.2. *Properties of the coefficient $\langle k\rangle$*

In this section, λ_i, ρ_i and C_i are taken constant on Ω_i. The signs of $\langle k\rangle_1$ and $\langle k\rangle_2$ are easily determined from the variational equation (17) with W and θ equal to k

$$\int_{\Omega_2}(\lambda_2'\nabla_y k\nabla_y\bar{k}+\mathrm{i}\omega\rho_2 C_2 k\bar{k})\,\mathrm{d}\Omega=\int_{\Omega_2}\rho_2 C_2\,\mathrm{i}\omega\bar{k}\,\mathrm{d}\Omega.$$

Then

$$\int_{\Omega_2}\rho_2 C_2 k_1\,\mathrm{d}\Omega=\int_{\Omega_2}\rho_2 C_2 k\bar{k}\,\mathrm{d}\Omega\geqslant 0, \tag{24}$$
$$\int_{\Omega_2}\rho_2 C_2\omega k_2\,\mathrm{d}\Omega=\int_{\Omega_2}\lambda_2'\nabla_y k\nabla_y\bar{k}\,\mathrm{d}\Omega\geqslant 0. \tag{25}$$

We obtain

$$\langle k\rangle_1\geqslant 0,\quad \langle k\rangle_2\geqslant 0.$$

Moreover, using the Schwartz inequality and equation (24),

$$\left|\int_{\Omega_2}k\,\mathrm{d}\Omega\right|=|\Omega|\,|\langle k\rangle|\leqslant|\Omega_2|^{1/2}\|k\|_{L^2}$$
$$=|\Omega_2|^{1/2}(|\Omega|\langle k\rangle_1)^{1/2},$$

and then

$$n^{1/2}\langle k\rangle_1^{1/2}\geqslant|\langle k\rangle|\geqslant\langle k\rangle_1,\quad \langle k\rangle_1\leqslant n.$$

Since we have also $|\langle k\rangle|\geqslant\langle k\rangle_2:\langle k\rangle_2\leqslant n$,

then

$$0\leqslant\langle k\rangle_1\leqslant n,$$
$$0\leqslant\langle k\rangle_2\leqslant n.$$

Changing ω into $-\omega$ in the complex conjugate of the variational form (17) leads to

$$\int_{\Omega_2}(\lambda_2'\nabla_y\bar{W}\nabla_y\theta+\mathrm{i}\omega\rho_2 C_2\bar{W}\theta)\,\mathrm{d}\Omega$$
$$=-\int_{\Omega_2}\rho_2 C_2\,\mathrm{i}\omega T^0\theta\,\mathrm{d}\Omega,\quad \forall\theta\in\mathscr{W},$$

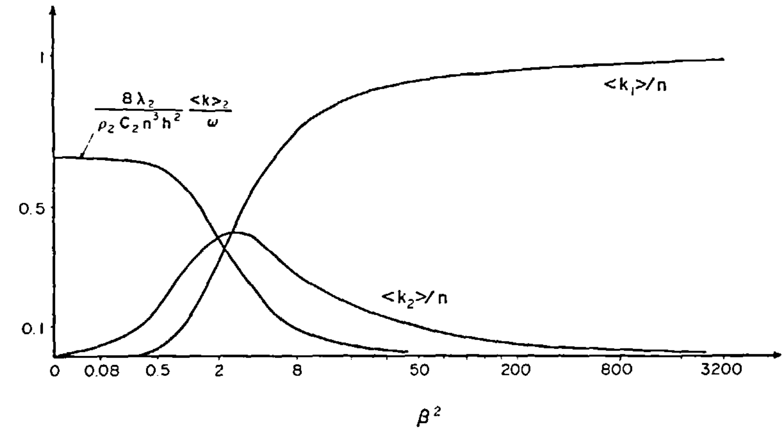

FIG. 3. Bilaminated composite: real and imaginary parts of coefficient $\langle k\rangle$ vs dimensionless pulsation β^2.

which is identical to equation (17). Therefore $\bar{k}(-\omega) = k(\omega)$.

Real and imaginary parts of $\langle k\rangle$ are respectively even and odd. As $\omega \mapsto 0$ we can see from equations (24) and (25) that $k_1 = O(\omega^2), k_2 = O(\omega)$ and so $k_2/\omega = O(1)$. All coefficients of equation (23) are of order 1 in magnitude.

4.3. *Examples*

Let us consider again the case of Section 3.4 (Fig. 2), where we now take

$$\lambda_2 = \varepsilon^2\lambda_2', \quad \lambda_2' = O(1).$$

The coefficient $\langle k\rangle$ is easily calculated by

$$\langle k\rangle = n\left(1-\frac{\tanh(\mathrm{i}^{1/2}\beta)}{\mathrm{i}^{1/2}\beta}\right),$$

$$\beta = \left(\frac{\omega\rho_2 C_2}{\lambda_2}\right)^{1/2}\frac{nh}{2}.$$

The parameter β represents a dimensionless pulsation. The real and imaginary parts of $\langle k\rangle/n$ are plotted on Fig. 3 vs β. Figure 4 shows, for this case, the memory function $\hat{K}(t)$ vs the dimensionless time τ

$$\hat{K}(t) = 8n\sum_{p=0}^{\infty}\frac{\exp[-(2p+1)^2\pi^2\tau/4]}{(8p+1)^2\pi^2},$$

$$\tau = \frac{4\lambda_2 t}{\rho_2 C_2 n^2 h^2}.$$

Looking at the tensor $\langle\lambda_1\rangle$, it is clear that

$$\langle\lambda_1\rangle_{22} = \langle\lambda_1\rangle_{33} = (1-n)\lambda_1, \quad \langle\lambda_1\rangle_{11} = 0.$$

For a macroscopic gradient directed along the perpendicular to the layers, the macroscopic description of Section 4.2 is not valid and scattering occurs. The process of homogenization used in this paper is inefficient because the pulsation ω has high values.

For circular cylindrical inclusion of radius a, or spherical inclusions of radius a, the results for $\langle k\rangle$ are,

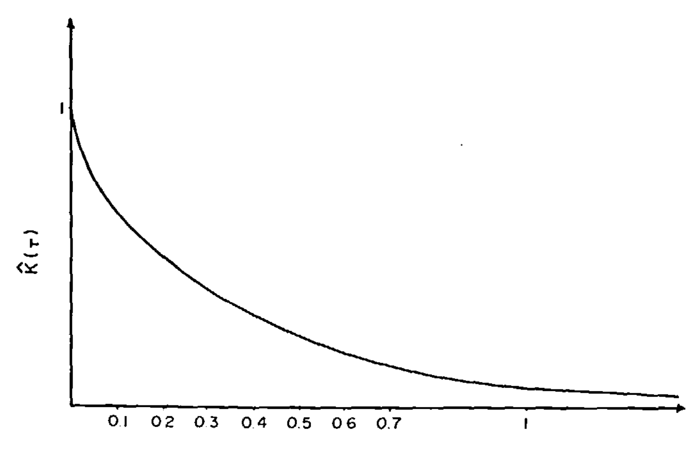

FIG. 4. Bilaminated composite: memory function $\hat{K}$ vs dimensionless time τ.

respectively,

$$\langle k\rangle_c = -nJ_2\left(\left(\frac{-i\omega}{\alpha}\right)^{1/2}a\right)\Big/J_0\left(\left(\frac{-i\omega}{\alpha}\right)^{1/2}a\right),$$

$$\langle k\rangle_s = n\left(1-\frac{3\cosh\left(\frac{i\omega}{\alpha}\right)^{1/2}a}{a\left(\frac{i\omega}{\alpha}\right)^{1/2}\sinh\left(\frac{i\omega}{\alpha}\right)^{1/2}a}+\frac{3\alpha}{i\omega a^2}\right),$$

with

$$\alpha = \frac{\lambda_2}{\rho_2 C_2}$$

and J_2, J_0 the Bessel functions of order 2 and 0. We have plotted on Fig. 5 the real and imaginary parts of $\langle k\rangle_c/n$ and $\langle k\rangle_s/n$ against the dimensionless pulsation $\omega a^2/\alpha$.

5. A SPECIAL CASE WITH A THREE COMPONENTS MEDIUM

We consider here a three component periodic system composed of two different heat conductors 1 and 2 with conductivities and capacities $O(1)$ separated by a third one with the lowest conductivity $O(\varepsilon^2)$ and capacity $O(1)$

$$\lambda_1 = O(1), \qquad \lambda_2 = O(1),$$
$$\lambda_3 = \lambda_3'\varepsilon^2, \qquad \lambda_3' = O(1).$$

Let n_1, n_2, n_3 denote the partial volumes of media 1, 2 and 3, and Γ_{13}, Γ_{23} their mutual boundaries. The equations for the temperature are, as above,

$$\nabla(\lambda_i \nabla T_i) = \rho_i C_i\, i\omega T_i, \quad i = 1,2,3 \tag{26}$$

with

$$[T]_\Gamma \doteq 0, \tag{27}$$

$$[\lambda \nabla T]_\Gamma \cdot \mathbf{n} = 0 \tag{28}$$

on the limit between two media.

Following the same process as in the preceding sections we obtain for the first order

$$T_1^0 = T_1^0(\mathbf{x}), \quad T_2^0 = T_2^0(\mathbf{x}),$$

with *a priori* $T_1^0 \neq T_2^0$ since there is no contact between media 1 and 2. A second order study then yields T_1^1 and T_2^1 in the form

$$T_1^1 = t_i^1 \nabla_{x_i} T_1^0 + \bar{T}_1^1(\mathbf{x}),$$
$$T_2^1 = t_i^2 \nabla_{x_i} T_2^0 + \bar{T}_2^2(\mathbf{x}),$$

where t_i^1 and t_i^2 are particular solutions for T_1^1 and T_2^1 corresponding respectively to

$$\nabla_{x_j} T_1^0 = \delta_{ij}, \quad \nabla_{x_j} T_2^0 = \delta_{ij},$$

and $\bar{T}_1^1(\mathbf{x})$, $\bar{T}_2^2(\mathbf{x})$ are arbitrary functions of the macroscopic variable $\mathbf{x}$.

Temperature T_3^0 is, on the other hand, given by

$$\nabla_y(\lambda_3^1 \nabla_y T_3^0) = \rho_3 C_3\, i\omega T_3^0, \tag{29}$$

where T_3^0 is Ω-periodic, with $T_3^0 = T_1^0(\mathbf{x})$ on the boundary Γ_{13} and $T_3^0 = T_2^0(\mathbf{x})$ on the boundary Γ_{23}. The solution is linear in T_1^0 and T_2^0 and can be put in the form

$$T_3^0 = f_1(\omega)T_1^0 + f_2(\omega)T_2^0. \tag{30}$$

The functions f_i, which depend on ω are complex valued.

Equation (26) for the order ε^0 and equation (28) for the order ε give, for media 1 and 2,

$$\nabla_y(\lambda_i(\nabla_y T_i^2 + \nabla_x T_i^1)) + \nabla_x(\lambda_i(\nabla_x T_i^0 + \nabla_y T_i^1)) = \rho_i C_i\, i\omega T_i^0 \quad i = 1,2 \tag{31}$$

and

$$\lambda_i(\nabla_y T_i^2 + \nabla_x T_i^1)\cdot \mathbf{n} = \lambda_3' \nabla_y T_3^0 \mathbf{n} \quad i = 1,2 \tag{32}$$

on the corresponding limits Γ_{i3}.

Since RHS of equation (32) can now be determined from equation (30), it appears that equations (31) must

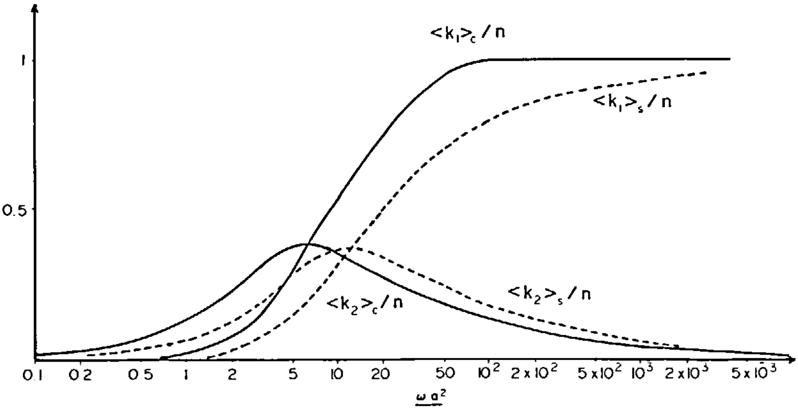

FIG. 5. Real and imaginary parts of coefficient $\langle k\rangle_c$ and $\langle k\rangle_s$ vs dimensionless pulsation for respectively circular cylindrical and spherical inclusions.

be considered separately. Two compatibility conditions are therefore obtained, taking the mean value of both sides in equations (31), respectively, on the volumes occupied by media 1 and 2

$$\nabla_x(\langle\lambda_1\rangle\nabla_x T_1^0) + A_{12}(\omega)T_2^0 + A_{11}(\omega)T_1^0 = \langle\rho_1 C_1\rangle\, i\omega T_1^0, \quad (33)$$

$$\nabla_x(\langle\lambda_2\rangle\nabla_x T_2^0) + A_{21}(\omega)T_1^0 + A_{22}(\omega)T_2^0 = \langle\rho_2 C_2\rangle\, i\omega T_2^0 \quad (34)$$

where

$$\langle\lambda_1\rangle = \frac{1}{|\Omega|}\int_{\Omega_1}\lambda_1(\delta+\nabla t^1)\,d\Omega,$$

$$\langle\lambda_2\rangle = \frac{1}{|\Omega|}\int_{\Omega_2}\lambda_2(\delta+\nabla t^2)\,d\Omega,$$

$$\langle\rho_1 C_1\rangle = \frac{1}{|\Omega|}\int_{\Omega_1}\rho_1 C_1\,d\Omega,$$

$$\langle\rho_2 C_2\rangle = \frac{1}{|\Omega|}\int_{\Omega_2}\rho_2 C_2\,d\Omega,$$

and where A_{ij} are complex valued and pulsation-dependent functions defined by

$$A_{ij} = \frac{-1}{|\Omega|}\int_{\Gamma_{i3}}\lambda_3'\nabla_y f_j \mathbf{n}_3\,dS$$

with $\mathbf{n}_3$ the unit exterior normal to medium 3. Moreover, the coupling between equations (33) and (34) is symmetrical as can be shown by taking equation (29) with $T_1^0 = 1$, $T_2^0 = 0$, multiplying the two members by f_2 and integrating over Ω_3,

$$\int_{\Omega_3} f_2\nabla_y(\lambda_3'\nabla_y f_1)\,d\Omega = \int_{\Gamma_{23}}\lambda_3'\nabla_y f_1 \mathbf{n}_3\,dS - \int_{\Omega_3}\lambda_3'\nabla_y f_2\nabla_y f_1\,d\Omega = \int_{\Omega_3}\rho_3 C_3\, i\omega f_1 f_2\,d\Omega.$$

Then, proceeding similarly for $T_2^0 = 1$, $T_1^0 = 0$ and f_2 gives

$$\int_{\Omega_3} f_2\nabla_y(\lambda_3'\nabla_y f_1)\,d\Omega = \int_{\Gamma_{13}}\lambda_3'\nabla_y f_2 \mathbf{n}_3\,dS - \int_{\Omega_3}\lambda_3'\nabla_y f_2\nabla_y f_1\,d\Omega = \int_{\Omega_3}\rho_3 C_3\, i\omega f_1 f_2\,d\Omega.$$

From which it results that

$$A_{12} = A_{21}.$$

For the case under consideration, the macroscopic behaviour is described by *two* temperature fields T_1^0 and T_2^0, defined at each point, determined by *two* symmetrically coupled differential equations, in contrast to the single temperature field and equation used in the classical treatment.

In the case of transient heat transfer, the coefficients A_{ij} must be replaced as in an above section by convolution operators. On the other hand, as ω tends to zero, the boundary value for T_3^0 is

$$T_3^0 = T_1^0(\mathbf{x}) = T_2^0(\mathbf{x}) = T^0(\mathbf{x}),$$

and the macroscopic behaviour is described by only one equation of classical structure

$$\nabla_x(\langle\lambda\rangle\nabla_x T^0) = 0,$$

with

$$\langle\lambda\rangle = \langle\lambda_1\rangle + \langle\lambda_2\rangle.$$

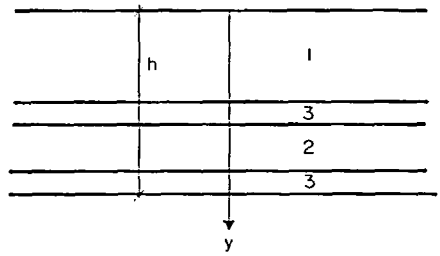

FIG. 6. Trilaminated composite.

As an example, let us consider the trilaminated medium shown (Fig. 6). The conductivities and capacities of media 1 and 2 are scalar constants of order 1, just as the capacity of medium 3. On the other hand, conductivity of the latter is of $O(\varepsilon^2)$. The length of the period is h. It is straightforward to obtain the values of the different coefficients in equations (33) and (34): for heat flux along the layers

$$\langle\lambda_1\rangle = n_1\lambda_2, \quad \langle\lambda_2\rangle = n_2\lambda_2,$$

$$A_{12} = A_{21} = \frac{2\lambda_3}{h}\left(\frac{i\omega}{\alpha}\right)^{1/2},$$

$$A_{11} = A_{22} = -\frac{2\lambda_3}{h}\left(\frac{i\omega}{\alpha}\right)^{1/2} \operatorname{cotanh}\left(\frac{i\omega}{\alpha}\right)^{1/2}\frac{n_3 h}{2}$$

with

$$\alpha = \frac{\lambda_3}{\rho_3 C_3}.$$

6. CONCLUDING REMARKS

The work presented here shows that many different situations can occur which induce different structures for the macroscopic behaviour. Each case must be studied separately, introducing small parameters only when necessary.

For simplicity only two or three components composites with or without strong discontinuities in their characteristics have been considered. The homogenization process obviously applies to more complex geometries (for instance, more components or composites with small inclusions: $n_i \ll 1$) providing that the periodicity is preserved.

It is clear that the procedure fails when considering frequencies which are too high. In such cases, the heat transfer equation for each constituent has the form of equation (12) and the solution cannot be represented by expansion (1). In fact, such an asymptotic development is valid only if the wavelength is large compared to the dimension l of the period Ω.

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DESCRIPTION MACROSCOPIQUE DE LA CONDUCTION THERMIQUE DANS LES COMPOSITES PERIODIQUES

Résumé—On étudie le comportement macroscopique d'un composite à structure périodique fine pour l'écoulement statique ou transitoire de la chaleur. La longueur d'onde du phénomène est supposée grande par rapport à la dimension de la période. Différentes situations sont envisagées au moyen de la méthode d'homogénéisation; elles conduisent à diverses structures pour la description macroscopique: équation unique aux dérivées partielles (classique), avec mémoire, système d'équations avec mémoire etc. On présente quelques exemples simples où le calcul analytique est possible.

EFFEKTIVE MAKROSKOPISCHE BESCHREIBUNG DER WÄRMELEITUNG IN PERIODISCH ZUSAMMENGESETZTEN STRUKTUREN

Zusammenfassung—Effektive Parameter des makroskopischen Verhaltens periodisch zusammengesetzter Strukturen werden bezüglich der stationären und instationären Wärmeleitung ermittelt, und zwar für den Fall, daß die Wellenlänge des Vorgangs groß im Vergleich zur Länge der Periode der Anordnung ist. Mit Hilfe der Homogenisierungsmethode werden verschiedene Situationen untersucht. Sie führen auf unterschiedliche Formen der makroskopischen Beschreibung: einzelne klassische partielle Differentialgleichung, einzelne partielle Integral/ Differentialgleichung mit Gedächtnis, System partieller Differentialgleichungen, usw.... Einige einfache Beispiele die sich analytisch lösen lassen, werden angegeben.

ЭФФЕКТИВНОЕ МАКРОСКОПИЧЕСКОЕ ОПИСАНИЕ ТЕПЛОПРОВОДНОСТИ В ПЕРИОДИЧЕСКИХ КОМПОЗИЦИОННЫХ СРЕДАХ

Аннотация—Определены эффективные параметры макроскопического поведения (статическая или неустановившаяся теплопроводность) периодических композиционных сред в случае, когда длина волны велика по сравнению с длиной периода. Используя метод сглаживания, проведен анализ различных ситуаций и получены различные макроскопические описания: одно классическое дифференциальное уравнение в частных производных, одно интегрально-дифференциальное уравнение в частных производных, учитывающее эффект памяти, система дифференциальных уравнений в частных производных и др. Приведено несколько простых примеров, когда можно получить аналитические результаты.